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LETTER TO THE EDITOR

Superuniversal spectral dimension for dilute branched polymers?

Muhammad Sahimi and Gary R Jerauld

Department of Chemical Engineering and Materials Science, University of Minnesota, Minneapolis, Minnesota 55455, USA

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Abstract. We introduce the problem of random walks on lattice animals (dilute branched polymers), and calculate the spectral dimension d_s of animals in two and three dimensions by using a two-parameter position-space renormalisation group method. This problem, which is a generalisation of the problem of 'the ant in the labyrinth', has been independently proposed and investigated by Wilke *et al.* The spectral dimension is given by $d_s = 2d_a/d_w$ where d_a is the fractal dimension of the lattice animal, and d_w the fractal dimension of the random walk on the lattice animal. Our results indicate that d_s may be a superuniversal quantity, i.e. its value is independent of dimension. Moreover, we find d_s to be close to $\frac{4}{3}$ which is the spectral dimension of the largest percolation cluster at the percolation threshold p_c as conjectured by Alexander and Orbach. Since both lattice animals and the largest percolation cluster at p_c have a *homogeneous interior structure*, our results suggest that the spectral dimension of *all* such fractals may equal $\frac{4}{3}$.

The problem of 'the ant in the labyrinth' was introduced by de Gennes (1976) in an attempt to probe the structure of percolation clusters (for a review see Mitescu and Roussenq (1983)). In this problem one considers the motion of a particle ('the ant') which performs a Pólya random walk on the occupied sites or bonds of a percolation network ('the labyrinth'); a Pólya walk is an unbiased, nearest-neighbour random walk. For such a walk, the root-mean-squared displacement R of the random walker is related to the number of steps N_w of the walk through the relation

$$R \sim N_w^{\nu_w} \tag{1}$$

where ν_w is a constant. A fractal dimension d_w for the random walk (RW) is defined by $d_w = 1/\nu_w$. Above the percolation threshold p_c and at very long times, $d_w = 2$ at all dimensions, whereas below p_c there are only finite clusters of occupied sites or bonds and thus R remains finite. At p_c the largest cluster is a fractal object whose radius of gyration ξ_p is related to the 'mass' N of the fractal through the fractal dimension

$$\xi_p \sim N^{1/d_p}. \tag{2}$$

The fractal dimension d_p is a dimensional-dependent quantity whose exact value is not known at any dimension. However, as a result of the den Nijs–Pearson–Nienhuis *et al* conjectures (den Nijs 1979, Pearson 1980, Nienhuis *et al* 1980) there is growing evidence that $d_p = \frac{91}{48} \simeq 1.896$ is possibly exact in two dimensions (2D) (see the review by Sahimi (1983)). For RWS on percolation clusters slightly above p_c the fractal dimension d_w remains unchanged if $R \gg \xi_p$. However, if $R \ll \xi_p$, then d_w takes values

that depend on the dimensionality of the system; no exact results have been found for d_w.

Alexander and Orbach (1982) made a remarkable discovery. They defined a new fractal dimension d_s by the relation

$$d_s = 2d_f/d_w, \tag{3}$$

which they called fracton or spectral dimension and conjectured, based on numerical evidence, that d_s is independent of dimension. They proposed that

$$d_s = \tfrac{4}{3}. \tag{4}$$

Recent numerical simulations (Derrida and Vannimenus 1982, Derrida *et al* 1983, Sahimi *et al* 1983, Pandey and Stauffer 1983, Mitescu and Musolf 1983) strongly supported their conjecture.

The Alexander–Orbach (AO) conjecture motivates an interesting question: is there any other fractal whose spectral dimension d_s is a *superuniversal* quantity, i.e. is independent of the details and dimensionality of the system? If there is such a fractal, one is motivated to know whether d_s of such a fractal is the same as that of the largest percolation cluster at p_c. In this letter we introduce the problem of RWS on lattice animals, i.e. connected clusters of sites or bonds in a d-dimensional lattice. This is a generalisation of 'the ant in the labyrinth' problem and has been independently proposed and investigated by Wilke *et al* (1984). The problem is interesting, because lattice animals are fractals with homogeneous interior structure and thus one may calculate their spectral dimension. Moreover, as a very accurate estimate of the fractal dimension d_a of animals in 2D, $d_a \simeq 1.56$ (Derrida and De Seze 1982) and its exact values in 3D, $d_a = 2$ (Parisi and Sourlas 1981) and 4D, $d_a = \frac{12}{5}$ (Dhar 1983) are known, the spectral dimension d_s of lattice animals may be determined to a high degree of accuracy. In this letter we report some preliminary calculations of d_s for lattice animals in 2D and 3D, using a two-parameter position-space renormalisation group (PSRG) method, which indicate that the spectral dimension d_s of lattice animals may be a superuniversal quantity, whose value is close to and possibly the same as that of the largest percolation cluster at p_c. In a future paper we will report results of Monte Carlo simulations of the problem of RWS on lattice animals. Because the statistics of branched polymers in the dilute limit in a good solvent are in the same universality class as those of lattice animals (Lubensky and Isaacson 1979), our results suggest a superuniversal spectral dimension for dilute branched polymers.

Before proceeding to describe our PSRG treatment of the problem, we point out that at very long times one expects that $R \sim \xi_a$, where ξ_a is the radius of the animal. The RW is diffusive in this regime. However, for shorter (intermediate) times we expect to have $R \ll \xi_a$ and thus a fractal dimension d_w should describe the RW on the animal in this case. We also note that the upper critical dimension d_c of lattice animals is $d_c = 8$ (Lubensky and Isaacson 1979). At d_c the fractal dimension d_a of animals is $d_a = 4$, the same as that of the largest percolation cluster at p_c at d_c (percolation) $= 6$. Although we do not have any rigorous proof, it is reasonable to assume that d_w of a RW on lattice animals at $d_c = 8$ is the same as that of the walk at p_c at d_c (percolation) $= 6$, i.e. $d_w = 6$. Therefore the mean-field value of the spectral dimension of animals is presumably $\frac{4}{3}$, the same as the AO conjecture.

To treat the present problem with a PSRG method, one has to develop a PSRG method for lattice animals. To this end we use the PSRG method of Family (1980, 1983) for lattice animals. Since in the lattice animal problem one is interested in the

statistics of all distinct clusters starting at the origin of an infinite lattice, one only rescales a renormalisation cell if it contains a single cluster originating at a fixed origin on the cell. In addition, one has to develop connectivity rules for rescaling an animal within a cell into a rescaled cell. We consider *site* animals, i.e. connected clusters of sites originating at a single site. For these animals we define several connectivity rules (see Family 1983, Reynolds *et al* 1978). In rule r_0 a cell is rescaled to an occupied site if it contains a single connected cluster that, starting from a fixed origin (which for all rules we choose to be the lower-left corner of the cell, as shown in figure 1), extends in any of the d possible directions across the cell in d dimensions; whereas rule r_1 requires a site animal to span in *one* specific direction. Finally rule r_2 requires a site animal to span in *all* directions. We consider all these rules and expect them to converge to the same results in the large cell limit.

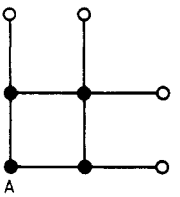

Figure 1. A $b=2$ cell that was used in the PSRG calculations of lattice animals and RWs on lattice animals. A denotes the origin of the animals and the RWs. The full circles are the sites of the animals.

To construct a renormalisation group (RG) transformation for site animals we first assign a fugacity S to each site in an animal and then determine the generating function (Family 1983)

$$G_i(S, b) = \sum_n C_i(n) S^n, \tag{5}$$

where $C_i(n)$ is the total number of site animals with n sites spanning according to the rule r_i $(i=0, 1, 2)$ on a cell of linear dimension b. The RG transformation is defined by requiring that the generating function for the spanning animals is invariant on the original and rescaled levels. This leads us to an equation for the renormalised fugacity S' on the rescaled cell; for the rule r_i S' is given by (Family 1983)

$$S' = G_i(S, b). \tag{6}$$

The fractal dimension d_a of animals is given by $d_a = \ln \lambda_a / \ln b$, where $\lambda_a = (\partial S'/\partial S)_{S^*}$ is the eigenvalue of the linearised RG transformation. S^* is the fixed point of the RG transformation, i.e. the solution of the equation $S^* = G_i(S^*, b)$.

After generating a spanning animal on the cell, a walker ('the ant') performs a RW on the spanning animal, starting from the origin of the animal. Of course the origin of the RW can be chosen arbitrarily. At each site of the animal there are j ways to take the next step of the RW with equal probabilities. We thus assign a fugacity $W = W_1/j$ to each *step* of the RW where W_1 is the *total* fugacity of the j ways of performing the RW (Nakanishi and Family 1984). The RW is considered spanning if the walker spans the cell in *any* of the possible directions. Because the RW can be performed along only those bonds whose end sites are part of the spanning animal,

the recursion relation for W', the renormalised RW fugacity, is given by

$$S'^2 W' = \sum_n S^n \left(\sum_M D_i(m) W^m \right). \tag{7}$$

The term S^n reflects the fact that a spanning animal of n sites has a fugacity S^n. $D_i(m)$ is the number of spanning RWs of m steps for the rule r_i. However, there are an infinite number of spanning RWs on any spanning animal; thus (7) cannot be evaluated exactly. But from (1) we see that only those spanning RWs contribute significantly to (7) at the fixed point W^* whose number of steps m satisfies $m \leqslant R^{d_w}$. This procedure has been used very recently for some RW problems with considerable success (Gould *et al* 1983, Sahimi and Jerauld 1983a, b, Sahimi *et al* 1984). Of course we do not know the value of d_w for RWs on lattice animals. But from the obvious inequalities $d_s > 1$ and $d_w > d_a$ we obtain

$$d_a < d_w < 2d_a. \tag{8}$$

Thus $1.56 < d_w < 3.12$ in 2D and the mid-point of this interval is about 2.3. Thus we enumerated only those walks whose number of steps m satisfied $m \leqslant R^{2.3}$, where R is the end-to-end length of the spanning walk on the cell. For the cells used in this paper, $R = [(d-1)(b-1)^2 + b^2]^{1/2}$, where d is the dimension and b the linear dimension of the cell.

Of course, we expect that by using larger and larger cells the results with any rule for the enumeration of spanning RWs converge to the same result. The mid-point of the interval was chosen to facilitate the convergence of the small cell results to asymptotic ones. To check the accuracy of the procedure we also enumerated shorter and longer spanning RWs. The results did not change significantly, which is expected since the fixed point W^* is a small number in the interval $(0, 1)$. For example, if we enumerate walks of up to m steps with $m \leqslant R^2$ in 2D, after extrapolating the results (see below) they differ by about 6% from the results with $m \leqslant R^{2.3}$. Thus we believe that the results are representative. In 3D $2 < d_w < 4$ and therefore we enumerated walks of up to $m \leqslant R^3$ steps. The recursion relations for S' and W' were determined for cells of size $b = 2$–4 on a square lattice using rules r_0, r_1 and r_2. In 3D it is not possible to calculate, in closed form, the recursion relation for S' for cells of linear dimension $b > 2$. The fractal dimension of RWs is given by $d_w = \ln \lambda_w / \ln b$, where the eigenvalue λ_w is given by $\lambda_w = (\partial W'/\partial W)_{W^*, S^*}$.

The results for W^*, S^*, d_w and d_a with rules r_0, r_1 and r_2 are given in table 1 for RWs on 2D lattice animals. Our results on the $b = 2$ cell with different rules in 3D are presented in table 2. In many previous PSRG studies (Reynolds *et al* 1980, Eschbach *et al* 1981, Family and Reynolds 1981) the finite b results were extrapolated to $b \to \infty$ by the following equation:

$$d_a(b) = d_a + a_1 (\ln b)^{-1} + a_2 (\ln b)^{-2}, \tag{9}$$

where a_1 and a_2 are some constants. However, the results of Stauffer (1981) and Tsallis (1982) indicate that a better (more accurate) method of extrapolating the finite b results is by using the equation

$$d_a(b) = d_a + (c_1 + c_2 b^{-\Omega d_a})(\ln b)^{-1}, \tag{10}$$

where c_1 and c_2 are again some constants. Here Ω is a correction-to-scaling exponent which is believed to be universal. For lattice animals in 2D Guttman (1982) obtained

Table 1. The fixed points S^* and W^* and the fractal dimensions d_a and d_w for lattice animals and RWs on lattice animals on a square lattice, using (*a*) rule r_0, (*b*) rule r_1, and (*c*) rule r_2. b is the linear dimension of the cell.

(*a*) Rule r_0.

b	S^*	W^*	d_a	d_w
2	0.3247	0.7316	1.254	1.694
3	0.3330	0.8674	1.344	1.951
4	0.3236	0.9183	1.387	2.022

(*b*) Rule r_1.

b	S^*	W^*	d_a	d_w
2	0.4142	0.6998	1.409	1.654
3	0.3754	0.8338	1.431	1.875
4	0.3487	0.8962	1.447	1.946

(*c*) Rule r_2.

b	S^*	W^*	d_a	d_w
2	0.5321	0.9112	1.655	1.901
3	0.4363	0.8918	1.603	2.045
4	0.3869	0.9231	1.581	2.074

Table 2. The fixed points S^* and W^* and the fractal dimensions d_a and d_w for lattice animals and RWs on lattice animals on a simple cubic lattice using rules r_0, r_1 and r_2. The results are for a $b=2$ cell.

Rule	S^*	W^*	d_a	d_w
r_0	0.1824	0.6496	1.404	1.820
r_1	0.2452	0.5926	1.722	1.675
r_2	0.3280	0.7889	2.182	2.354

$\Omega = 0.87 \pm 0.06$, whereas more recently Margolina *et al* (1983) obtained $\Omega = 0.86 \pm 0.05$. By using the data of table 1 and equation (9) we found $d_a \simeq 1.60$, 1.54 and 1.48 for rules r_0, r_1 and r_2 respectively. We took Ω to be 0.85 and used (10) to obtain $d_a \simeq 1.637$, 1.575 and 1.466 for rules r_0, r_1 and r_2 respectively. The latter results are precisely 2.2% different from the former ones which shows that (9) is reliable, although (10) has a better theoretical basis. If we take Ω to be 0.8, the results virtually do not change. Since the value of Ω for the RW problem is not known, we used (9) to extrapolate our results for d_w. We obtained $d_w \simeq 2.20$, 2.18 and 2.08 for rules r_0, r_1 and r_2 respectively. Because asymptotically these rules are expected to converge to the same result, we combine the three sets of data and find the value of $d_a(d_w)$ by finding the value of the intercept which gives the best overall fit to the three sets of data simultaneously. From this procedure we find $d_w \simeq 2.15$ and $d_a \simeq 1.53$ (by using (9)) and $d_a \simeq 1.55$ (by using (10)). The results for d_a agree will with the estimate of

d_a given by Derrida and De Seze (1982), $d_a \simeq 1.56$. Thus the spectral dimension $d_s = 2d_a/d_w$ of lattice animals (dilute branched polymers) in 2D is found to be about

$$d_s(d=2) \simeq 1.42. \tag{11}$$

This is only 6.5% larger than $d_s \simeq \frac{4}{3}$ of the AO conjecture for the largest percolation cluster at p_c. Moreover, if we use $d_w \simeq 2.20$ for rule r_0 we obtain $d_s \simeq 1.39$. In 3D our results are not very accurate since we used only a $b=2$ cell. However, from table 1 it appears that the trends in the results with rules r_0, r_1 and r_2 for d_a and d_w are consistent with each other, i.e. a large value of d_a (e.g. $d_a \simeq 1.6$ with rule r_0) also means a large value of d_w (e.g. $d_w \simeq 2.20$ with rule r_0). It is possible to obtain rough estimates of d_a and d_w in 3D by using our PSRG results in 2D and 3D. Let us define $f(b) = d_a/d_a(b)$, where d_a is the correct value of this exponent and $d_a(b)$ is its value obtained from PSRG calculations with a cell of size b using rule r_i ($i=0, 1, 2$). From table 2 it appears that the results with rule r_2 are more accurate with a small cell in 3D; this is also true of the results in 2D. Thus if we use the results of rule r_2 in 2D we obtain $f(2) \simeq 0.93$. If we assume that $f(b)$ is independent of dimensions we obtain $d_a(d=3) \simeq 0.93 \times 2.182 = 2.03$, where we used the result with rule r_2 in 3D. Our estimate $d_a \simeq 2.03$ is in fact very close to the exact result $d_a = 2$ (Parisi and Sourlas 1981); thus this procedure seems to be reliable. If we use the same procedure for d_w we obtain the estimate $d_w \simeq 2.66$ in 3D. Thus a rough estimate of the spectral dimension d_s of lattice animals in 3D is given by

$$d_s(d=3) \simeq 1.52. \tag{12}$$

Despite several approximations made in obtaining this estimate, it differs by only about 7% from our estimate of d_s in 2D. We believe if one uses larger renormalisation cells, the agreement will improve. Wilke *et al* (1984) have performed Monte Carlo simulations and have obtained $d_w(d=2) \simeq 2.6 \pm 0.3$ and $d_w(d=3) \simeq 3.4 \pm 0.4$, which then result in $d_s(d=2) \simeq 1.2 \pm 0.15$ and $d_s(d=3) \simeq 1.18 \pm 0.14$, in reasonable agreement with our results.

In summary, we have introduced and investigated the problem of RWS on lattice animals. By developing a two-parameter PSRG transformation we estimated the spectral dimension d_s of lattice animals (dilute branched polymers) in 2D and 3D. Although we only used small renormalisation cells, our results for d_s in both 2D and 3D agreed with each other and were in reasonable agreement with the value $d_s = \frac{4}{3}$ conjectured by Alexander and Orbach (1982) for the largest percolation cluster at p_c. Moreover, the value of d_s for lattice animals at the upper critical dimensionality $d_c = 8$ appears to be exactly $\frac{4}{3}$. These results suggest that the spectral dimension of lattice animals (dilute branched polymers) is a superuniversal quantity and is the same as that of the largest percolation clusters at p_c. Thus although percolation clusters and lattice animals belong to two different universality classes (Lubensky and Isaacson 1979), they may be related to each other through the concept of spectral dimension. We also note that if the spectral dimension of lattice animals is indeed superuniversal and equal to $\frac{4}{3}$, a Flory approximation for d_w can be obtained. By using the Flory approximation for d_a (Isaacson and Lubensky 1980, Daoud and Joanny 1981), $d_a = (2d+4)/5$, we obtain a Flory approximation for d_w:

$$d_w = (3d+6)/5. \tag{13}$$

Our result may also indicate that the spectral dimension for *all fractals* which have *homogeneous interior structure* may be the same and equal $\frac{4}{3}$. We will report the results of our Monte Carlo simulations in a future paper.

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